Convex Optimization

Part 1: Preliminaries

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Linear algebra

Vector $x \in \mathbb{R}^n$

- $x = (x_1, ..., x_n)$
- length and direction (*e.g.*, $n = 2$)
- column vectors, row vectors

Linear algebra

p-norm of vector $x \in \mathbb{R}^n$ where $1 \leq p \leq \infty$:

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} .$$

- $p = 1$
- $p = 2$
- $p = \infty$
- Example of $v = (3, 4)$

Linear algebra

Subspace

The subset $\mathcal{S} \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if the following property holds: If x and y are any two elements of $\mathcal{S}$, then

$$\alpha x + \beta y \in \mathcal{S}, \quad \forall \alpha, \beta \in \mathbb{R} \ .$$

(*i.e.*, set closed under addition and scaling)

Linear algebra

Span

$\{s_1, s_2, ..., s_k\}$ is a spanning set for $\mathcal{S}$ if any vector $s \in \mathcal{S}$ can be written as

$$s = \alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_k s_k \ ,$$

for some $\alpha_1, \alpha_2, ..., \alpha_k$.

Linear independence

A set of vectors $x_1, x_2, ..., x_k \in \mathbb{R}^n$ is called linearly independent if there are no $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = 0 \ ,$$

except $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$.

(*i.e.*, $x_1, x_2, ..., x_k$ are linearly independent if none of them can be written as a linear combination of the others.)

Linear algebra

Basis

If $\{x_1, ..., x_k\}$ are linearly independent & span $\mathcal{X}$, we call them a basis of $\mathcal{X}$

- k (the number of elements in the basis) is referred to as the dimension of $\mathcal{X}$, and denoted by $\dim(\mathcal{X})$.
- There are many ways to choose a basis of $\mathcal{X}$ in general, but that all bases contain the same nubmer of vectors.

Linear algebra

Inner product / dot product on $\mathbb{R}^n$

$$u \cdot v = \langle u, v \rangle = u^\top v = \sum_{i=1}^{n} u_i v_i$$

Linear algebra

Angle between u and v

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|_2\|v\|_2}$$

- If they are perpenicular, $\langle u, v\rangle = 0$.

Linear algebra

Projection of v onto u

$$v_{\mathsf{p}} = \frac{\langle v, u \rangle}{\|u\|_2^2} u$$

Linear algebra

Cauchy-Schwarz inequality

$$|u^\top v| \leq \|u\|_2 \|v\|_2$$

- Two sides are equal iff u and v are linearly dependent.

Linear algebra

Triangle inequality

$$\|u + v\|_2 \leq \|u\|_2 + \|v\|_2$$

Linear algebra

Outer product

$$u \otimes v = uv^\top = \begin{bmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_n \\ u_2v_1 & u_2v_2 & \dots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \dots & u_nv_n \end{bmatrix}$$

Linear algebra

Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix}$$

Some concepts to recall

- square matrix
- transpose of a matrix
- symmetric matrix

Linear algebra

Null $(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

Range $(A) = \{y \in \mathbb{R}^n : Ax = y \text{ for some } x\}$

Rank $(A) =$ dimension of span of columns/rows of A

Linear algebra

If A is $n \times n$, Rank $(A) = n$ iff

- Null $(A) = \{0\}$
- Range $(A) = \mathbb{R}^n$
- det $(A) \neq 0$

Linear algebra

For a matrix $A \in \mathbb{R}^{n \times n}$, an eigenvalue λ and eigenvector v are those that satisfy

$$Av = \lambda v \ .$$

Linear algebra

For a symmetric matrix A,

- All eigenvalues are real.
- All eigenvectors are perpendicular to each other.

If A is nonsingular, none of its eigenvalues are zero.

Linear algebra

For a symmetric matrix A, eigen or spectral decomposition

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^\top \ ,$$

or

$$A = Q \Lambda Q^\top \ ,$$

using matrix forms.

Linear algebra

Positive semidefinite matrix

- A symmetric matrix A is called positive semidefinite, if all eigenvalues are greater than or equal to 0.
- Or
$$x^\top Ax \geq 0 \ , \quad \forall x \in \mathbb{R}^n$$
- $AA^\top$ and $A^\top A$ are always psd.

Analysis

Interior

An element $x \in C \subseteq \mathbb{R}^n$ is called an interior point of C if there exists an $\epsilon > 0$ for which

$$\{y \mid \|y - x\|_2 \leq \epsilon\} \subseteq C \ ,$$

i.e., there exists a ball centered at x that lies entirely in C.

Int C: the set of all points interior to C.

Analysis

Supremum ("least upper bound") and infimum ("greatest lower bound")

Suppose $C \subseteq \mathbb{R}$. A number a is an upper bound on C if $x \leq a, \ \forall x \in C$.

The set of upper bounds on C is either

1. empty (C is unbounded above)
2. all of $\mathbb{R}$ ($C = \emptyset$)
3. a closed infinite interval $[b, \infty)$

Then, supremum of C (or $\sup C$) becomes

1. ∞
2. $-\infty$
3. b

If the set C is finite, $\sup C$ is the maximum of its elements.

Calculus

Functions and derivatives

- Continuity
- Differentiability
- Derivative
- Gradient
- Hessian
- Quadratic function example

Any questions?