Convex Optimization

Part 4: Duality I

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POSTECH

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Admin

Remaining courseworks

- Assignments 3, 4, 5
- Quiz 2
- Final exam

Some references this course heavily relies on include

- Convex Optimization, Stephen Boyd and Lieven Vandenberghe
- Convex Optimization: Algorithms and Complexity, Sébastien Bubeck
- Convex Optimization, Ryan Tibshirani
- Optimization Algorithms, Constantine Caramanis
- and more (see cvxopt website)

Standard form problem

Optimization problem in the standard form (not necessarily convex)

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- optimization variable $x \in \mathbb{R}^n$
- objective function $f_0 : \mathbb{R}^n \to \mathbb{R}$
- inequality constraint functions $f_i : \mathbb{R}^n \to \mathbb{R}$
- equality constraint functions $h_i : \mathbb{R}^n \to \mathbb{R}$
- domain $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \ \cap \ \bigcap_{i=1}^{p} \operatorname{dom} h_i$
- optimal value $p^\star$

Expressing problems in standard form

Box constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i \leq x_i \leq u_i, \quad i = 1, \dots, n \end{array}$$

Standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i \leq 0, \quad i = 1, \dots, 2n \end{array}$$

where

$$f_i = \begin{cases} l_i - x_i & \text{for } i = 1, \dots, n \\ x_{i-n} - u_{i-n} & \text{for } i = n+1, \dots, 2n \end{cases}$$

Lagrangian

Augment the objective function with a weighted sum of the constraint functions

Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ with $\operatorname{dom} L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- λ_i is Lagrange multiplier associated with $f_i(x) \leq 0$
- ν_i is Lagrange multiplier associated with $h_i(x) = 0$
- can be interpreted as soft linear approximation of hard indicator functions

Lagrange dual function

Minimum value of the Lagrangian over x

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- a concave function even when the standard form problem is not convex because it is pointwise minimum of affine functions of λ, ν
- can be $-\infty$ when the Lagrangian is unbounded below in x

Lower bounds on optimal value

Lower bound property:

$$g(\lambda, \nu) \leq p^\star \qquad \text{for any } \lambda \succeq 0 \text{ and } \nu$$

Proof.

For $\tilde{x}$ a feasible point for the problem, we have

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x})}_{\leq 0} \leq f_0(\tilde{x})$$

Thus

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

The proof is finished by noting that this holds for every feasible point $\tilde{x}$ □

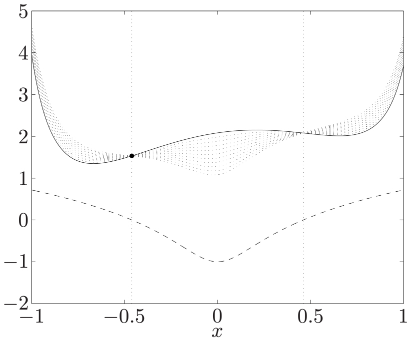

Figure: Lower bound from a dual feasible point; figure from BV

- (solid curve) objective function f_0
- (dashed curve) constraint function f_1
- (dotted vertical lines) feasible set $[-0.46, 0.46]$
- (circle) $x^\star = -0.46$, $p^\star = 1.54$
- (dotted curves) $L(x, \lambda)$ for $\lambda = 0.1, \ldots, 1.0$

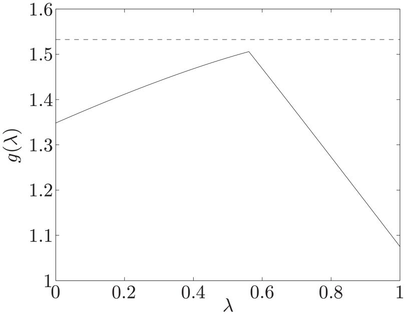

Figure: Dual function g; figure from BV

- (dashed line) optimal value $p^\star$
- (solid line) dual function $g(\lambda)$
- (x-axis) dual variable λ

f_0 or f_1 not necessarily convex, but g is concave

Least norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^\top x \\ \text{subject to} & Ax = b \end{array}$$

- Lagrangian

$$L(x, \nu) = x^\top x + \nu^\top (Ax - b)$$

- Dual function

$$g(\nu) = \inf_x L(x, \nu) = L(-\frac{1}{2}A^\top \nu, \nu) = -\frac{1}{4}\nu^\top A A^\top \nu - b^\top \nu$$

  which is a concave quadratic function of ν

- Lower bound property

$$p^\star \geq -\frac{1}{4}\nu^\top A A^\top \nu - b^\top \nu \qquad \text{for any } \nu$$

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

- Lagrangian

$$\begin{aligned} L(x, \lambda, \nu) &= c^\top x - \lambda^\top x + \nu^\top (Ax - b) \\ &= -b^\top \nu + (c + A^\top \nu - \lambda)^\top x \end{aligned}$$

- Dual function

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

  g is linear on affine domain $\mathbf{dom}\, g = \{(\lambda, \nu) \mid A^\top \nu - \lambda + c = 0\}$, hence concave

- Lower bound property

$$p^\star \geq -b^\top \nu \quad \text{if } A^\top \nu + c \succeq 0$$

Any questions?