Convex Optimization

Part 4: Duality II

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Assignment 3

- will be out this Friday
- based on coding
- due in 2 weeks

Lagrange dual problem

What is the best lower bound that can be obtained from the Lagrange dual function?

Lagrange dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- a convex optimization problem (max concave objective with convex constraint; regardless of whether or not the primal problem is convex)
- optimal value denoted by $d^\star$ ($d^\star = -\infty$ if problem is infeasible; $d^\star = +\infty$ if unbounded above)
- a pair (λ, ν) is dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \operatorname{dom} g$; and $(\lambda^\star, \nu^\star)$ dual optimal

Making dual constraints explicit

Domain of the dual function

$$\operatorname{dom} g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}$$

- often $\operatorname{dom} g$ has dimension smaller than $m + p$; some equality constraints are hidden or implicit in g
- can form an equivalent problem in which these equality constraints are given explicitly as constraints

Lagrange dual of standard form LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

$$\rightarrow \quad \begin{array}{ll} \text{maximize} & -b^\top \nu \\ \text{subject to} & A^\top \nu - \lambda + c = 0 \\ & \lambda \succeq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \text{maximize} & -b^\top \nu \\ \text{subject to} & A^\top \nu + c \succeq 0 \end{array}$$

- equivalent Lagrange dual problems (but not the same)

Lagrange dual of inequality form LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \preceq b \end{array} \qquad \ldots \qquad \begin{array}{ll} \text{maximize} & -b^\top \lambda \\ \text{subject to} & A^\top \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}$$

- reformulated by explicitly including the dual feasibility conditions as constraints
- symmetry between the standard and inequality form LPs and their duals; the dual of a standard form LP is an LP with only inequality constraints, and vice versa
- can show that dual of the inequality form LP is (equivalent to) the primal problem

Weak duality

Weak duality

$$d^\star \leq p^\star$$

- holds even if the original problem is not convex
- can be used to find a nontrivial lower bound; useful for difficult problems

Optimal duality gap

$$p^\star - d^\star \geq 0$$

- always nonnegative

Duality gap

Duality gap: with primal feasible x and dual feasible (λ, ν)

$$f(x) - g(\lambda, \nu)$$

is called the duality gap

From the lower bound property, we have

$$f(x) - p^\star \leq f(x) - g(\lambda, \nu)$$

- this also indicates that when the duality gap is zero, the primal is optimal
- can provide a stopping criterion of iterative methods

Strong duality

Strong duality

$$d^\star = p^\star$$

- zero optimal duality gap; Lagrange dual function is tight
- does not hold in general; usually (but not always) holds for convex problems
- there exist sufficient conditions beyond convexity that guarantee strong duality which are called constraint qualifications

Slater's condition

Slater's condition: given the primal problem is convex, if the problem is strictly feasible, *i.e.*, if there exists an $x \in \operatorname{int} \mathcal{D}$ such that

$$f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b$$

then strong duality holds

- guarantees strong duality: $p^\star = d^\star$
- also guarantees that the dual optimal value is attained when $d^\star > -\infty$
- there exists many other types of constraint qualifications

Refinement of Slater's condition

When the first k constraint functions $f_1, \ldots, f_k$ are affine, strong duality holds under the following weaker condition: if there exists an $x \in \operatorname{int} \mathcal{D}$ with

$$\underbrace{f_i(x) \leq 0, \quad i = 1, \ldots, k,}_{\text{no strict inequality}} \qquad f_i(x) < 0, \quad i = k+1, \ldots, m, \qquad Ax = b$$

then strong duality holds

- in other words, the affine inequalities do not need to hold with strict inequality
- Slater's condition reduces to feasibility when the constraints are all linear equalities and inequalities, and $\operatorname{dom} f_0$ is open

Least norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^\top x \\ \text{subject to} & Ax = b \end{array} \qquad \text{maximize} \quad -\frac{1}{4}\nu^\top A A^\top \nu - b^\top \nu$$

- the dual problem is an unconstrained concave quadratic maximization problem
- Slater's condition is simply that the primal problem is feasible; then $p^\star = d^\star$ strong duality holds

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \preceq b \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^\top \lambda \\ \text{subject to} & A^\top \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}$$

- from Slater's condition: $p^\star = d^\star$ if $A\tilde{x} \prec b$ for some $\tilde{x}$; *i.e.*, strong duality holds if the problem is feasible
- in fact this holds for any LP either standard or inequality form
- can confirm that this holds for the dual as well if it is feasible
- thus $p^\star = d^\star$ except when both the primal and dual problems are infeasible $(p^\star = \infty, d^\star = -\infty)$

Any questions?