Convex Optimization Part 4: Duality II

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#### Admin

Assignment 3

- will be out this Friday
- based on coding
- due in 2 weeks

### Lagrange dual problem

What is the best lower bound that can be obtained from the Lagrange dual function?

Lagrange dual problem

 $\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$ 

- a convex optimization problem (max concave objective with convex constraint; regardless of whether or not the primal problem is convex)
- ▶ optimal value denoted by d<sup>\*</sup> (d<sup>\*</sup> = -∞ if problem is infeasible; d<sup>\*</sup> = +∞ if unbounded above)
- ▶ a pair  $(\lambda, \nu)$  is dual feasible if  $\lambda \succeq 0$ ,  $(\lambda, \nu) \in \text{dom } g$ ; and  $(\lambda^*, \nu^*)$  dual optimal

## Making dual constraints explicit

Domain of the dual function

$$\operatorname{dom} g = \{(\lambda,\nu) ~|~ g(\lambda,\nu) > -\infty\}$$

- ▶ often dom g has dimension smaller than m + p; some equality constraints are hidden or implicit in g
- can form an equivalent problem in which these equality constraints are given explicitly as constraints

## Lagrange dual of standard form LP

equivalent Lagrange dual problems (but not the same)

## Lagrange dual of inequality form LP

minimize 
$$c^{\top}x$$
 ... maximize  $-b^{\top}\lambda$   
subject to  $Ax \leq b$  subject to  $A^{\top}\lambda + c = 0$   
 $\lambda \geq 0$ 

reformulated by explicitly including the dual feasibility conditions as constraints

- symmetry between the standard and inequality form LPs and their duals; the dual of a standard form LP is an LP with only inequality constraints, and vice versa
- ▶ can show that dual of the inequality form LP is (equivalent to) the primal problem

### Weak duality

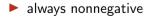
#### Weak duality

$$d^\star \le p^\star$$

- holds even if the original problem is not convex
- > can be used to find a nontrivial lower bound; useful for difficult problems

Optimal duality gap

$$p^{\star} - d^{\star} \ge 0$$



## Duality gap

Duality gap: with primal feasible x and dual feasible  $(\lambda, \nu)$ 

 $f(x) - g(\lambda, \nu)$ 

is called the duality gap

From the lower bound property, we have

$$f(x) - p^* \le f(x) - g(\lambda, \nu)$$

this also indicates that when the duality gap is zero, the primal is optimal
can provide a stopping criterion of iterative methods

## Strong duality

#### Strong duality

$$d^{\star} = p^{\star}$$

- zero optimal duality gap; Lagrange dual function is tight
- does not hold in general; usually (but not always) holds for convex problems
- there exist sufficient conditions beyond convexity that guarantee strong duality which are called constraint qualifications

#### Slater's condition

**Slater's condition**: given the primal problem is convex, if the problem is strictly feasible, *i.e.*, if there exists an  $x \in int \mathcal{D}$  such that

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

then strong duality holds

- ▶ guarantees strong duality:  $p^{\star} = d^{\star}$
- $\blacktriangleright$  also guarantees that the dual optimal value is attained when  $d^{\star}>-\infty$
- there exists many other types of constraint qualifications

#### Refinement of Slater's condition

When the first k constraint functions  $f_1, \ldots, f_k$  are affine, strong duality holds under the following weaker condition: if there exists an  $x \in \text{int } \mathcal{D}$  with

$$\underbrace{f_i(x) \le 0, \quad i = 1, \dots, k,}_{\text{no strict inequality}} \qquad f_i(x) < 0, \quad i = k + 1, \dots, m, \qquad Ax = b$$

then strong duality holds

- ▶ in other words, the affine inequalities do not need to hold with strict inequality
- Slater's condition reduces to feasibility when the constraints are all linear equalities and inequalities, and dom f<sub>0</sub> is open

#### Least norm solution of linear equations

minimize 
$$x^{\top}x$$
 maximize  $-\frac{1}{4}\nu^{\top}AA^{\top}\nu - b^{\top}\nu$   
subject to  $Ax = b$ 

the dual problem is an unconstrained concave quadratic maximization problem
Slater's condition is simply that the primal problem is feasible; then p\* = d\* strong duality holds

## Inequality form LP

minimize
$$c^{\top}x$$
maximize $-b^{\top}\lambda$ subject to $Ax \preceq b$ subject to $A^{\top}\lambda + c = 0$  $\lambda \succeq 0$ 

- From Slater's condition:  $p^* = d^*$  if  $A\tilde{x} \prec b$  for some  $\tilde{x}$ ; *i.e.*, strong duality holds if the problem is feasible
- ▶ in fact this holds for any LP either standard or inequality form
- can confirm that this holds for the dual as well if it is feasible
- ► thus p<sup>\*</sup> = d<sup>\*</sup> except when both the primal and dual problems are infeasible (p<sup>\*</sup> = ∞, d<sup>\*</sup> = -∞)

# Any questions?