CSED700H: Convex Optimization

Convex functions[1]

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POSTECH

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[1] slides credits to Prof. Lieven Vandenberghe

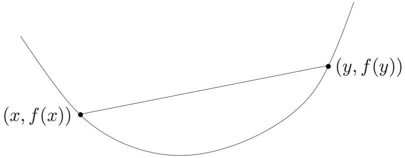
Contents

Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

  for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 \leq \theta \leq 1$

Examples on $\mathbb{R}$

Convex

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- exponential: e^{ax}, for any $a \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

Concave

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b, \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$

Examples on $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

Examples on $\mathbb{R}^n$

- affine function $f(x) = a^\top x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Examples on $\mathbb{R}^{m\times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathrm{tr}(A^\top X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- 2-norm (spectral norm): maximum singular value

$$f(X) = \|X\|_2 = \sigma_{\mathsf{max}}(X) = (\lambda_{\mathsf{max}}(X^\top X))^{1/2}$$

Restriction of a convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbb{R}^n$

can check convexity of f by checking convexity of functions of one variable

Example: $f : \mathbb{S}^n \to \mathbb{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, f = \mathbb{S}^n_{++}$

$$\begin{aligned} g(t) = \log\det(X + tV) &= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log\det X + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$

g is concave in t (for any choice of $X \succ 0$, V); hence f is concave

Extended-value extension

extended-value extension $\tilde{f}$ of f is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom}\, f, \qquad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom}\, f$$

often simplifies notation; for example, the condition

$$0 \le \theta \le 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbb{R} \cup \{\infty\}$), means the same as the two conditions

- $\mathbf{dom}\, f$ is convex
- for $x, y \in \mathbf{dom}\, f$,

$$0 \le \theta \le 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$$

First-order condition

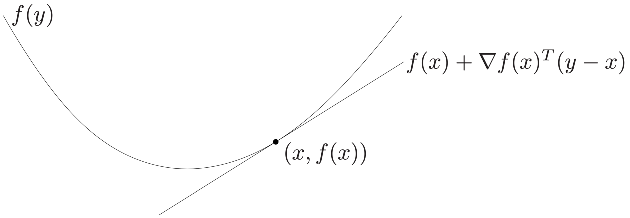
f is **differentiable** if $\mathrm{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathrm{dom}\, f$

First-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) \quad \text{for all } x, y \in \mathrm{dom}\, f$$



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\mathrm{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbb{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathrm{dom}\, f$

Second-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathrm{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x\, \mathrm{dom}\, f$, then f is strictly convex

Examples

Quadratic function: $f(x) = (1/2)x^\top Px + q^\top x + r$ (with $P \in \mathbb{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

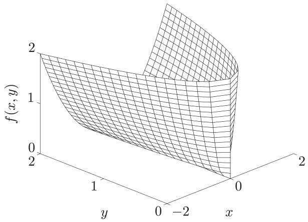
Least squares objective: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^\top(Ax - b), \qquad \nabla^2 f(x) = 2A^\top A$$

convex (for any A)

Quadratic-over-linear function: $f(x, y) = x^2/y$

$$\nabla^2 f(x,y) = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y \\ -x \end{bmatrix}^\top \succeq 0$$



convex for $y > 0$

Examples

Log-sum-exp function: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^\top z} \operatorname{diag}(z) - \frac{1}{(\mathbf{1}^\top z)^2} z z^\top \qquad \text{with } z_k = \exp x_k$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^\top \nabla^2 f(x) v \geq 0$ for all v:

$$v^\top \nabla^2 f(x) v = \frac{(\sum_{k=1}^{n} z_k v_k^2)(\sum_{k=1}^{n} z_k) - (\sum_{k=1}^{n} v_k z_k)^2}{(\sum_{k=1}^{n} z_k)^2} \geq 0$$

since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy-Schwarz inequality)

Geometric mean: $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbb{R}_{++}^n$ is concave
(similar proof as for log-sum-exp)

Epigraph and sublevel set

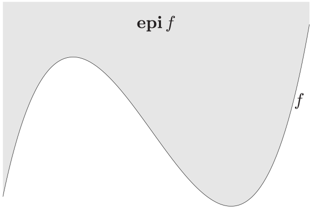
α-**sublevel set** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathbb{C}_\alpha = \{x \in \operatorname{dom} f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

Epigraph of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\operatorname{epi} f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \operatorname{dom} f,\ f(x) \leq t\}$$



f is convex if and only if $\operatorname{epi} f$ is a convex set

Jensen's inequality

Basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

Extension: if f is convex, then

$$f(\mathbb{E}z) \leq \mathbb{E}f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathrm{prob}(z = x) = \theta, \qquad \mathrm{prob}(z = y) = 1 - \theta$$

Operations that preserve convexity

methods for establishing convexity of a function

1. very definition (often simplified by restricting to a line)
2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
3. show that f is obtained from simple convex functions by operations that preserve convexity
    - nonnegative weighted sum
    - composition with affine function
    - pointwise maximum and supremum
    - composition
    - minimization
    - perspective

Positive weighted sum and composition with affine function

Nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

Sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

Composition with affine function: $f(Ax + b)$ is convex if f is convex

Examples

- logarithmic barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^\top x), \qquad \mathbf{dom}\, f = \{x \mid a_i^\top x < b_i,\ i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if $f_1, \dots, f_m$ are convex, then

$$f(x) = \max\{f_1(x), \dots, f_m(x)\}$$

is convex

Examples

- piecewise-linear function: $f(x) = \max_{i=1,\dots,m}(a_i^\top x + b_i)$ is convex
- sum of r largest components of $x \in \mathbb{R}^n$:

  $$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

  is convex ($x_{[i]}$ is ith largest component of x)

  proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

Examples

- *support function* of a set: $\mathbb{S}_{\mathbb{C}}(x) = \sup_{y \in \mathbb{C}} y^\top x$ is convex for any set $\mathbb{C}$
- distance to farthest point in a set $\mathbb{C}$:

$$f(x) = \sup_{y \in \mathbb{C}} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbb{S}^n$

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^\top X y$$

Composition with scalar functions

composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$:

$$f(x) = h(g(x))$$

$$f \text{ is convex if } \begin{cases} g \text{ convex}, \ h \text{ convex}, \ \tilde{h} \text{ nondecreasing} \\ g \text{ concave}, \ h \text{ convex}, \ \tilde{h} \text{ nonincreasing} \end{cases}$$

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

Examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$$

$$f \text{ is convex if } \begin{cases} g_i \text{ convex}, \; h \text{ convex}, \; \tilde{h} \text{ nondecreasing in each argument} \\ g_i \text{ concave}, \; h \text{ convex}, \; \tilde{h} \text{ nonincreasing in each argument} \end{cases}$$

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^\top \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^\top g''(x)$$

Examples

- $\sum_{i=1}^m \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^m \log g_i(x)$ is convex if g_i are convex

Minimization

if $f(x, y)$ is convex in (x, y) and $\mathbb{C}$ is a convex set, then

$$g(x) = \inf_{y \in \mathbb{C}} f(x, y)$$

is convex

Examples

- $f(x, y) = x^\top Ax + 2x^\top By + y^\top Cy$ with

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^\top (A - BC^{-1}B^\top)x$
  g is convex, hence Schur complement $A - BC^{-1}B^\top \succeq 0$
- distance to a set: $d(x, \mathbb{S}) = \inf_{y \in \mathbb{S}} \|x - y\|$ is convex if $\mathbb{S}$ is convex

Perspective

the **perspective** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$g(x,t) = tf(x/t), \qquad \operatorname{dom} g = \{(x,t) \mid x/t \in \operatorname{dom} f,\ t > 0\}$$

g is convex if f is convex

Examples

- $f(x) = x^\top x$ is convex; hence $g(x,t) = x^\top x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy

  $$g(x,t) = t \log t - t \log x$$

  is convex on $\mathbb{R}^2_{++}$
- if f is convex, then

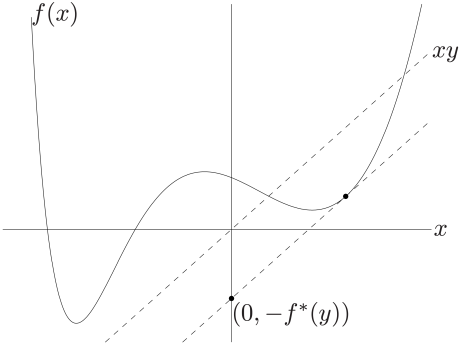
  $$g(x) = (c^\top x + d) f\big((Ax + b)/(c^\top x + d)\big)$$

  is convex on $\{x \mid c^\top x + d > 0,\ (Ax + b)/(c^\top x + d) \in \operatorname{dom} f\}$

The conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^\top x - f(x) \right)$$



- f^* is convex (even if f is not)
- will be useful in chapter 5

Examples

- negative logarithm $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0} (xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic $f(x) = (1/2)x^\top Qx$ with $Q \in \mathbb{S}^n_{++}$

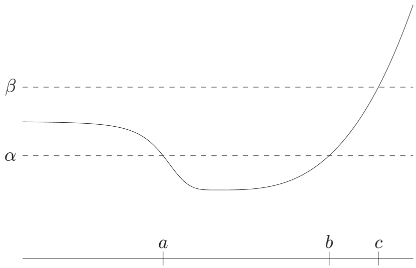
$$\begin{aligned} f^*(y) &= \sup_x (y^\top x - (1/2)x^\top Qx) \\ &= \frac{1}{2} y^\top Q^{-1} y \end{aligned}$$

Quasiconvex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $\operatorname{dom} f$ is convex and the sublevel sets

$$\mathbb{S}_\alpha = \{x \in \operatorname{dom} f \mid f(x) \le \alpha\}$$

are convex for all α



$$\begin{aligned} \mathbb{S}_\alpha &= [a, b] \\ \mathbb{S}_\beta &= (-\infty, c) \end{aligned}$$

- f is quasiconcave if $-f$ is quasiconvex
- f is quasilinear if it is quasiconvex and quasiconcave

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbb{R}$
- $\operatorname{ceil}(x) = \inf\{z \in \mathbb{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbb{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^\top x + b}{c^\top x + d}, \qquad \operatorname{dom} f = \{x \mid c^\top x + d > 0\}$$

- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \qquad \operatorname{dom} f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$$

is quasiconvex

Internal rate of return

- cash flow $x = (x_0, \ldots, x_n)$; x_i is payment in period i (to us if $x_i > 0$)
- we assume $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$
- *present value* of cash flow x, for interest rate r:

$$PV(x, r) = \sum_{i=0}^{n} (1+r)^{-i} x_i$$

- *internal rate of return* is smallest interest rate for which $PV(x, r) = 0$:

$$IRR(x) = \inf\{r \geq 0 \mid PV(x, r) = 0\}$$

IRR is quasiconcave: superlevel set is intersection of open halfspaces

$$IRR(x) \geq R \qquad \Longleftrightarrow \qquad \sum_{i=0}^{n} (1+r)^{-1} x_i > 0 \quad \text{for } 0 \leq r < R$$

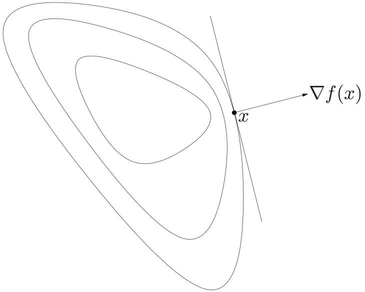
Properties

Modified Jensen inequality: for quasiconvex f

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

First-order condition: differentiable f with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^\top (y - x) \leq 0$$



Sums: sums of quasiconvex functions are not necessarily quasiconvex

Log-concave and log-convex functions

a positive function f is log-concave if $\log f$ is concave:

$$f(\theta x + (1-\theta)y) \geq f(x)^{\theta} f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

f is log-convex if $\log f$ is convex

- powers: x^a on $\mathbb{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, *e.g.*, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^{\top} \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function Φ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

Properties of log-concave functions

- twice differentiable f with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^\top \quad \text{for all } x \in \operatorname{dom} f$$

- product of log-concave functions is log-concave
- sum of log-concave functions is not always log-concave
- integration: if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave, then

$$g(x) = \int f(x, y)dy$$

  is log-concave (not easy to show)

Consequences of integration property

- convolution $f * g$ of log-concave functions f, g is log-concave

$$(f * g)(x) = \int f(x-y)g(y)dy$$

- if $\mathbb{C} \subset \mathbb{R}^n$ convex and y is a random variable with log-concave p.d.f. then

$$f(x) = \mathrm{prob}(x + y \in \mathbb{C})$$

is log-concave
proof: write $f(x)$ as integral of product of log-concave functions

$$f(x) = \int g(x+y)p(y)dy, \qquad g(u) = \begin{cases} 1 & u \in \mathbb{C} \\ 0 & u \notin \mathbb{C}, \end{cases}$$

p is p.d.f. of y

Example: yield function

$$Y(x) = \text{prob}(x + w \in \mathbb{S})$$

- $x \in \mathbb{R}^n$: nominal parameter values for product
- $w \in \mathbb{R}^n$: random variations of parameters in manufactured product
- $\mathbb{S}$: set of acceptable values

if $\mathbb{S}$ is convex and w has a log-concave p.d.f., then

- Y is log-concave
- yield regions $\{x \mid Y(x) \geq \alpha\}$ are convex