Online Convex Programming and Generalized Infinitesimal Gradient Ascent (ICML 2003)

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Introduction

- Convex programming consists of a convex feasible set $F \subset \mathbb{R}^n$ and a convex cost function $c : F \to \mathbb{R}$.
- Most of the methods and topics we discussed in-class were optimization for machine learning under an *offline environment*, where we have full access of the cost function, training data,..., beforehand.
- We discuss *online convex programming*, in which an algorithm faces a sequence of convex programming problems, each with the same feasible set but different cost functions.
- Each time the algorithm must choose a point before it observes the cost function.

Online Convex Programming

Definition

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Definition

At each time step t, an **online convex programming algorithm** selects a vector $x^t \in F$. After the vector is selected, it receives the cost function c^t.

Online Convex Programming

- In online convex programming, all information (e.g. cost functions c^t for all time step t) is not available before decisions are made.
- Therefore, online algorithms do not reach "solutions" (e.g. minima), but instead achieve certain goals.
- A measure of performance called *regret* is considered.
- *Average regret* is the regret divided by T, the number of rounds.

Online Convex Programming

The remainder of this paper is presented under the following assumptions.

1. The feasible set F is **bounded**.
   That is, there exists $N \in \mathbb{R}$ such that $\forall x, y \in F$, $\|x - y\|_2 \leq N$.
2. The feasible F is **closed**. That is, for all sequences $\{x^1, x^2, \ldots\}$ where $x^t \in F$ for all t, if there exists a $x \in \mathbb{R}^n$ such that $x = \lim_{t \to \infty} x^t$, then $x \in F$.
3. The feasible set F is **nonempty**.
4. For all t, c^t is **differentiable**[1].

[1] We can relax this assumption in terms of the existence of subgradient as follows: Given x, there exists a vector g_x such that $c^t(y) \geq c^t(x) + g_x(y - x)$ for all y.

Online Convex Programming

The remainder of this paper is presented under the following assumptions.

5. There exists an $N \in \mathbb{R}$ such that for all t, for all $x \in F$, $\|\nabla c^t(x)\|_2 \leq N$.
6. For all t, there exists an algorithm, given x, which produces $\nabla c^t(x)$.
7. For all $y \in \mathbb{R}^n$, there exists an algorithm which can produce the projection of y onto F defined as $\text{proj}_F(y) = \arg\min_{x \in F} \|x - y\|_2$.[1]

[1] This paper uses the notation $P(y)$, but we shall use this one since we did in-class.

Online Convex Programming

Algorithm 1 (*Greedy Projection*)

Select an arbitrary $x^1 \in F$ and a sequence of learning rates $\eta_1, \eta_2, ... \in \mathbb{R}^+$. In time step t, after receiving a cost function c^t, select the next vector x^{t+1} according to:

$$x^{t+1} = \text{proj}_F(x^t - \eta_t \nabla c^t(x^t)).$$

Online Convex Programming

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$$x^{t+1} = \text{proj}_F(x^t - \eta_t \nabla c^t(x^t)).$$

- Note that if we happened to know that cost functions c^t were actually all identical, i.e., $c^t = c$ for all time steps t, then *Greedy Projection* is exactly the same as PGD which we have learned in-class [s06-2].

Analyzing the Performance of the Algorithm

Definition

Given an algorithm A, and a convex programming problem $(F, \{c^1, c^2, ...\})$, if $\{x^1, x^2, ...,\}$ are the vectors selected by A, then the **cost** of A until time T is

$$C_A(T) = \sum_{t=1}^{T} c^t(x^t).$$

The cost of a static feasible solution $x \in F$ until time T is

$$C_x(T) = \sum_{t=1}^{T} c^t(x).$$

The **regret** of algorithm A until time T is

$$R_A(T) = C_A(T) - \min_{x \in F} C_x(T).$$

Analyzing the Performance of the Algorithm

Theorem (*Greedy Projection*'s regret)

If $\eta_t = 1/\sqrt{t}$, the regret of the Greedy Projection algorithm is

$$R_G(T) \leq \frac{\|F\|^2\sqrt{T}}{2} + (\sqrt{T} - \frac{1}{2})\|\nabla c\|_2^2$$

where $\|F\| := \max_{x,y \in F} \|x - y\|_2$ *and* $\|\nabla c\| := \max_{x \in F, t \in \{1,2,\ldots\}} \|\nabla c^t(x)\|_2$.

Analyzing the Performance of the Algorithm

Proof

- We begin with arbitrary $\{c^1, c^2, ...\}$.
  Running *Greedy Projection*, we obtain $\{x^1, x^2, ...\}$.
- Because c^t is convex, for all x:

$$c^t(x) \geq c^t(x^t) + (\nabla c^t(x^t)) \cdot (x - x^t).$$

- Set x^* to be a statically optimal vector, i.e., $x^* := \arg\min_{x \in F} C_x(T)$.
  Since $x^* \in F$, from the previous inequality we have

$$c^t(x^*) \geq c^t(x^t) + (\nabla c^t(x^t)) \cdot (x^* - x^t).$$

- subtract both sides from $c^t(x^t)$, we get

$$c^t(x^t) - c^t(x^*) \leq c^t(x^t) - \left(c^t(x^t) + (\nabla c^t(x^t)) \cdot (x^* - x^t)\right).$$

Analyzing the Performance of the Algorithm

Proof (Continued)

- Define linear function $g^t(x) := \nabla c^t(x^t) \cdot x$. If we were to change function c^t to function g^t, the behavior of the algorithm will still be the same ($\because \nabla g^t(x^t) = \nabla c^t(x^t)$).
  That is, we will select the same $\{x^1, x^2, ...\}$.
- Thus, we can rewrite the previous inequality

$$c^t(x^t) - c^t(x^*) \leq \cancel{c^t(x^t)} - \left(\cancel{c^t(x^t)} + (\nabla c^t(x^t)) \cdot (x^* - x^t)\right).$$

  as

$$c^t(x^t) - c^t(x^*) \leq g^t(x^t) - g^t(x^*) := (x^t - x^*) \cdot \nabla c^t(x^t).$$

- We will now bound the RHS of this inequality.

Analyzing the Performance of the Algorithm

Proof (Continued)

- Define for all t, $y^{t+1} := x^t - \eta_t \nabla c^t(x^t)$.
- Then, we can rewrite *Greedy Projection* as

$$x^{t+1} = \text{proj}_F(x^t - \eta_t \nabla c^t(x^t)) = \text{proj}_F(y^{t+1}).$$

- By definition of y^{t+1}, we have

$$(y^{t+1} - x^*)^2 = \left((x^t - x^*) - \eta_t \nabla c^t(x^t)\right)^2$$

$$= (x^t - x^*)^2 - 2\eta_t (x^t - x^*) \cdot \nabla c^t(x^t) + \eta_t^2 \|\nabla c^t(x^t)\|_2^2$$
$$\leq (x^t - x^*)^2 - 2\eta_t (x^t - x^*) \cdot \nabla c^t(x^t) + \eta_t^2 \|\nabla c\|_2^2.$$

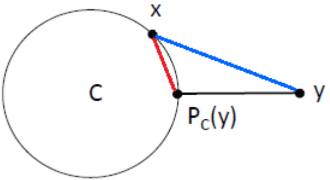
Analyzing the Performance of the Algorithm

Proof (Continued)

- Recall the following property of projection operator on convex sets, which we have discussed in-class.[1]

Property (Projection on a convex set F is contracting)

For all $y \in \mathbb{R}^n$, for all $x \in F$, $(\text{proj}_F(y) - x)^2 \leq (y - x)^2$.



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[1] A proof is given in Schneider, R. (2013). *Convex Bodies: The Brunn–Minkowski Theory*, page 9

[2] https://wikidocs.net/22434

Analyzing the Performance of the Algorithm

Proof (Continued)

- Using the previous inequality and this property, we have,

$$(x^{t+1} - x^*)^2 = (\mathrm{proj}_F(y^{t+1}) - x^*)^2 \leq (y^{t+1} - x^*)^2$$

$$\leq (x^t - x^*)^2 - 2\eta_t(x^t - x^*) \cdot \nabla c^t(x^t) + \eta_t^2 \|\nabla c\|_2^2.$$

- Rearranging terms and dividing both sides by $2\eta_t$, we get

$$(x^t - x^*) \cdot \nabla c^t(x^t) \leq \frac{1}{2\eta_t}\left((x^t - x^*)^2 - (x^{t+1} - x^*)^2\right) + \frac{\eta_t}{2}\|\nabla c\|_2^2.$$

- We conclude the following inequality, in which taking the summation from $t = 1, ..., T$ of the LHS will give regret $R_G(T)$.

$$c^t(x^t) - c^t(x^*) \leq \frac{1}{2\eta_t}\left((x^t - x^*)^2 - (x^{t+1} - x^*)^2\right) + \frac{\eta_t}{2}\|\nabla c\|_2^2.$$

Analyzing the Performance of the Algorithm

Proof (Continued)

- By summing we get

$$
\begin{aligned}
R_G(T) &\leq \sum_{t=1}^{T} \frac{1}{2\eta_t}\left((x^t - x^*)^2 - (x^{t+1} - x^*)^2\right) \\
&\quad + \frac{\eta_t}{2}\|\nabla c\|^2 \\
&\leq \frac{1}{2\eta_1}(x^1 - x^*)^2 - \frac{1}{2\eta_T}(x^{T+1} - x^*)^2 \\
&\quad + \frac{1}{2}\sum_{t=2}^{T}\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)(x^t - x^*)^2 \\
&\quad + \frac{\|\nabla c\|^2}{2}\sum_{t=1}^{T}\eta_t \\
&\leq \|F\|^2\left(\frac{1}{2\eta_1} + \frac{1}{2}\sum_{t=2}^{T}\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)\right) \\
&\quad + \frac{\|\nabla c\|^2}{2}\sum_{t=1}^{T}\eta_t \\
&\leq \|F\|^2\frac{1}{2\eta_T} + \frac{\|\nabla c\|^2}{2}\sum_{t=1}^{T}\eta_t
\end{aligned}
$$

- Now, if we define $\eta_t = 1/\sqrt{t}$

$$
\begin{aligned}
\sum_{t=1}^{T}\eta_t &= \sum_{t=1}^{T}\frac{1}{\sqrt{t}} \\
&\leq 1 + \int_{t=1}^{T}\frac{dt}{\sqrt{t}} \\
&\leq 1 + \left[2\sqrt{t}\right]_1^T \\
&\leq 2\sqrt{T} - 1
\end{aligned}
$$

- Plugging this to the previous inequality finishes the proof. □

Analyzing the Performance of the Algorithm

Theorem

If $\eta_t = 1/\sqrt{t}$, the regret of the Greedy Projection algorithm is

$$R_G(T) \leq \frac{\|F\|^2\sqrt{T}}{2} + (\sqrt{T} - \frac{1}{2})\|\nabla c\|_2^2$$

where $\|F\| := \max_{x,y \in F} \|x - y\|_2$ *and* $\|\nabla c\| := \max_{x \in F, t \in \{1,2,\ldots\}} \|\nabla c^t(x)\|_2$.

- Therefore, the average regret of *Greedy Projection* approaches to 0

$$\limsup_{T \to \infty} \frac{R_G(T)}{T} = 0.$$

- The first term of the bound is because we might begin on the wrong side of F.
- The second part is a result of the fact that we always respond (x^{t+1}) after we see the cost function (c^t).

Lazy Projection

- This section is only in the full version of the paper.[1]

Algorithm 2 (*Lazy Projection*)

Select an arbitrary $x^1 \in F$ and a sequence of learning rates $\eta_1, \eta_2, \ldots \in \mathbb{R}^+$. Define $y^1 = x^1$. In time step t, after receiving a cost function c^t, define y^{t+1}:

$$y^{t+1} = y^t - \eta_t \nabla c^t(x^t)$$

and select the vector

$$x^{t+1} = \text{proj}_F(y^{t+1}).$$

[1] *Online convex programming and generalized infinitesimal gradient ascent* (Technical Report CMU-CS-03-110). CMU

Lazy Projection

Theorem (*Lazy Projection*'s regret)

Given a constant learning rate η, Lazy Projection's regret is

$$R_L(T) \leq \frac{\|F\|^2}{2\eta} + \frac{\eta\|\nabla c\|^2 T}{2}.$$

The proof given is in the appendix of the full version of this paper.[1]

[1] *Online convex programming and generalized infinitesimal gradient ascent* (Technical Report CMU-CS-03-110). CMU

Greedy vs. Lazy

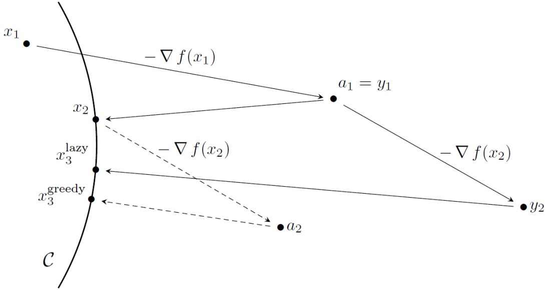

FIGURE 1. Graphical illustration of the greedy (dashed) and lazy (solid) branches of the projected subgradient (PSG) method. [1]

- *Greedy variant*: adds $-\nabla f(x_n)$ to x_n and projects back to $\mathcal{C}$ if needed.
- *Lazy variant*: the gradient term $-\nabla f(x_n)$ is **not** added to x_n, but to the "unprojected" iterate y_n. We only project to $\mathcal{C}$ in order to obtain the algorithm's next iterate.

[1] Kwon, J.,& Mertikopoulos, P. (2014). A continuous-time approach to online optimization. *Journal of Dynamics and Games*, 4(2):125–148, 2017

Conclusion and Relevant Papers

- **This paper presents an online form of the standard gradient descent from offline optimization**: Online Gradient Descent (OGD)
- This algorithm can guarantee $\mathcal{O}(\sqrt{T})$ regret for an arbitrary sequence of differentiable convex functions.
- **Note:** A sequence defined by algorithm A has "no-regret" if the regret is sublinear as a function of T, i.e., $R_A(T) = o(T)$.
- **How to improve the regret bound for strongly-convex losses?** E. Hazan, A. Agarwal, and S. Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69:169-192, 2007

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- Week 7-8: Study Online Convex Optimization (OCO) framework
- Week 9-10: Review recent papers related to OCO algorithms and its applications
- Week 11-12: Implement basic OCO algorithms via Pytorch/Tensorflow
- Week 13-14: Experiment regret convergence via datasets and apply OCO algorithms as optimizers for specific machine learning problems (e.g. SVM classification of MNIST dataset)
- Week 15: Project presentation

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